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DECOMPOSITION OF THE CONJUGACY REPRESENTATION OF THE SYMMETRIC GROUPS

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ABSTRACT

Consider the two natural representations of the symmetric group S_n on the group algebra $\mathbb{C}[S_n]$: the regular representation and the conjugacy representation (acting on the basis by conjugation). Let $m(\lambda)$ be the multiplicity of the irreducible representation S^λ in the conjugacy representation and let f^λ be the multiplicity of S^λ in the regular representation. By the character estimates of [R1] and [Wa] we prove

(1) For any $1 > \varepsilon > 0$ there exist $0 < \delta(\varepsilon)$ and $N(\varepsilon)$ such that, for any partition λ of $n > N(\varepsilon)$ with $\max\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}\} \le \delta(\varepsilon)$,

$$1 - \varepsilon < \frac{m(\lambda)}{f^\lambda} < 1 + \varepsilon$$

where λ_1 is the size of the largest part in λ and λ_1' is the number of parts in λ.

(2) For any fixed $1 > r > 0$ and $\varepsilon > 0$ there exist $k = k(\varepsilon, r)$ and $N(\varepsilon, r)$ such that, for any partition λ of $n > N(\varepsilon, r)$ with $\max\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}\} < r$,

$$A - \varepsilon < \frac{m(\lambda)}{f^\lambda} < A + \varepsilon$$

where A is a constant which depends only on the fractions

$$\frac{\lambda_1}{n}, \ldots, \frac{\lambda_k}{n}, \frac{\lambda_1'}{n}, \ldots, \frac{\lambda_k'}{n}.$$

This strengthens Adin–Frumkin's result [AF] and answers a question of Stanley [St].

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Introduction

There are two natural representations of any finite group G on the group algebra $\mathbb{C}[G] = \{\sum_{g\in G} a_g g | a_g \in \mathbb{C}\}$: the regular representation and the conjugacy representation. The **(left) regular representation** ρ is defined by (left) multiplication

$$\rho(h) \cdot \sum_{g\in G} a_g g = \sum_{g\in G} a_g hg.$$

The **conjugacy representation** ψ is defined by conjugation

$$\psi(h) \cdot \sum_{g\in G} a_g g = \sum_{g\in G} a_g hgh^{-1}.$$

Adin and Frumkin [AF] proved that in the case of the symmetric groups the corresponding characters are close. More precisely, the quotient of the norms of the regular character and the conjugacy character and the cosine of the angle between them tend to 1 when n tends to infinity. This implies that these representations have essentially the same decomposition.

Stanley suggested to apply the upper bound of [R1] for the study of the multiplicities of specific representations. This suggestion is based on the observation that the multiplicities of the irreducible representations in the conjugacy representation of any finite group are sums of characters. Combining the upper bound of [R1] and Wasserman's asymptotic formula [Wa] we obtain

THEOREM 2.1: *Let $m(\lambda)$ be the multiplicity of the irreducible representation S^λ in the conjugacy representation, and let f^λ be the multiplicity of S^λ in the regular representation.*

For any $1 > \varepsilon > 0$ there exist $0 < \delta(\varepsilon)$ and $N(\varepsilon)$ such that, for any partition λ of $n > N(\varepsilon)$ with $\max\{\frac{\lambda_1}{n}, \frac{\lambda'_1}{n}\} \leq \delta(\varepsilon)$,

$$1 - \varepsilon < \frac{m(\lambda)}{f^\lambda} < 1 + \varepsilon$$

where λ_1 is the size of the largest part in λ and λ'_1 is the number of parts in λ.

If $\max\{\frac{\lambda_1}{n}, \frac{\lambda'_1}{n}\}$ is bounded away from 1, then the multiplicity of S^λ in the conjugacy representation is proportional to f^λ as the next theorem shows.

THEOREM 2.2: *Let $m(\lambda)$ and f^λ be as in the above Theorem. For any fixed $1 > r > 0$ and $\varepsilon > 0$ there exist $k = k(\varepsilon, r)$ and $N(\varepsilon, r)$ such that, for any partition λ of $n > N(\varepsilon, r)$ with $\max\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}\} < r$,*

$$A - \varepsilon < \frac{m(\lambda)}{f^\lambda} < A + \varepsilon$$

where A is a constant which depends only on the fractions $\frac{\lambda_1}{n}, \dots, \frac{\lambda_k}{n}, \frac{\lambda_1'}{n}, \dots, \frac{\lambda_k'}{n}$, but not necessarily equals 1.

If $\max\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}\}$ is not bounded away from 1 then the fraction $m(\lambda)/f^\lambda$ may tend to infinity.

1. Preliminaries

1.1. THE CONJUGACY REPRESENTATION OF FINITE GROUPS. The study of the conjugacy representation begins with Frame [Fr]. The following elementary approach is due to Solomon [So] .

Let G be a finite group and let ρ be an irreducible representation. The multiplicity of ρ in a representation ϕ, $m(\rho, \phi)$, is given by

$$m(\rho, \phi) = \frac{1}{|G|} \sum_{g \in G} \chi^\rho(g) \chi^\phi(g)^*$$

where $\chi^\phi(g)^*$ is the complex conjugate of $\chi^\phi(g)$. See [Se] 2.3.

Let ψ be the conjugacy representation of G. Then

$$\chi^\psi(g) = |C_g| = \frac{|G|}{|C(g)|}$$

where C_g is the centralizer of g and $C(g)$ is the conjugacy class in G containing g.

So,

$$m(\rho, \psi) = \frac{1}{|G|} \sum_{g \in G} \chi^\rho(g) \chi^\psi(g)^* = \frac{1}{|G|} \sum_{g \in G} \chi^\rho(g) \frac{|G|}{|C(g)|} = \sum_C \chi^\rho(C)$$

where the last sum runs over all conjugacy classes C in G.

We conclude

LEMMA 1.1 ([So]): *For every finite group G and every irreducible representation ρ of G, the multiplicity of ρ in the conjugacy representation of G is $\sum_C \chi^\rho(C)$.*

This implies the following well known corollary

COROLLARY: *For every finite group G and every irreducible representation ρ of G the sum $\sum_C \chi^\rho(C)$ is a nonnegative integer.*

The following open problem is natural.

AN OPEN PROBLEM (Roth [Rt] – Saxl [Sa]): *For which finite groups is the sum $\sum_C \chi^\rho(C)$ a positive integer for all irreducible representations ? (in other words, all irreducible representations appear in the conjugacy representation).*

For partial solutions see [Fo], [K] and [L2].

A positive answer to this problem gives rise to the question whether the decompositions of the conjugacy representation and the regular representation are similar.

In [F] Frumkin proved that all irreducible representations appear in the conjugacy representation of the symmetric groups. See also [Sc] and [L1]. Adin and Frumkin [AF] proved that furthermore the regular representation and the conjugacy representation have essentially the same decomposition. Here we strengthen this result by considering Solomon's approach and applying recent estimates of characters.

1.2. CHARACTERS OF THE SYMMETRIC GROUPS. Let n be a positive integer. The partitions of n are designated by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda_k \le \cdots \le \lambda_2 \le \lambda_1$ are positive integers with $n = \lambda_1 + \cdots + \lambda_k$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ define the conjugate partition $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ by choosing λ'_i to be the number of parts of λ that are $\ge i$.

Let S_n be the symmetric group on n letters. Every partition λ of n has an associated irreducible representation S^λ of S_n over $\mathbb{C}$. Moreover, the set $\{S^\lambda |\ \lambda \text{ a partition of } n\}$ forms the complete list of irreducible representations of S_n over $\mathbb{C}$.

Let f^λ be the degree (= dimension) of S^λ, $\chi^\lambda(C)$ the character of S^λ at a conjugacy class C, and $r^\lambda(C)$ the normalized character of S^λ at C, i.e. $\frac{\chi^\lambda(C)}{f^\lambda}$. Note that the multiplicity of S^λ in the regular representation is f^λ.

Let C be a conjugacy class in S_n. It is well known that C consists of all permutations with the same cycle structure. The order of the support of C,

$\text{supp}(C)$, is the number of non-fixed digits under the action of a permutation in C.

In [R1] we prove the following upper bound for the normalized characters of S_n at any conjugacy class C.

THEOREM 1.2: *There exist constants $b > 0$ and $1 > q > 0$, such that for $n > 4$, for every conjugacy class C in the symmetric group S_n, and for every irreducible representation S^λ,*

$$|r^\lambda(C)| \leq \left(\max\left\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}, q\right\}\right)^{b\cdot\text{supp}(C)}$$

where $\text{supp}(C)$ is the order of the support of C, λ_1 is the size of the largest part of λ, and λ_1' is the number of parts in λ.

The following is an asymptotic formula for normalized characters of conjugacy classes with a bounded support. This result is due to Wasserman [Wa] and explained (and applied) in [FOW].

THEOREM 1.3: *Let m be a fixed positive integer, and let π be a permutation in S_m with γ_2 2-cycles, γ_3 3-cycles, etc.: π may be considered as an element of S_n for $n \geq m$. Let λ be a partition of n and set*

$$\alpha_i = \frac{\lambda_i - i + \frac{1}{2}}{n}, \quad \beta_i = \frac{\lambda_i' - i + \frac{1}{2}}{n}, \quad \text{and} \quad d = \max\{i|\ \lambda_i - i \geq 0\}$$

(i.e. length of the main diagonal in the Young diagram of λ). Then

$$r^\lambda(\pi) = \prod_{p\geq 2}\left(\sum_{i=1}^{d}[\alpha_i^p - (-\beta_i)^p]\right)^{\gamma_p} + O\left(\frac{1}{n}\right)$$

where the constant in $O(\frac{1}{n})$ depends only on π.

Let h_{ij} be the length of the (i,j)-hook in the Young diagram of λ. Obviously,

$$h_{ij} = \lambda_i - i + \lambda_j' - j + 1 \quad \text{and} \quad \sum_{i=1}^{d} h_{ij} = h,$$

where d is the length of the main diagonal in this Young diagram. The following elementary fact follows:

FACT 1.4:

$$\sum_{i=1}^{d}(\alpha_i + \beta_i) = \sum_{i=1}^{d}\left(\frac{\lambda_i - i + \frac{1}{2}}{n} + \frac{\lambda_i' - i + \frac{1}{2}}{n}\right) = \frac{1}{n}\sum_{i=1}^{d} h_{ii} = 1.$$

This fact will be useful in the proof of Theorem 2.2.

2. Proofs

In this section we prove Theorems 2.1 and 2.2. We make no attempt to optimize the constants for which they hold.

THEOREM 2.1: *Let $m(\lambda)$ be the multiplicity of the irreducible representation S^λ in the conjugacy representation, and let f^λ be the multiplicity of S^λ in the regular representation.*

For any $1 > \varepsilon > 0$ there exist $0 < \delta(\varepsilon)$ and $N(\varepsilon)$ such that, for any partition λ of $n > N(\varepsilon)$ with $\max\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}\} \leq \delta(\varepsilon)$,

$$1 - \varepsilon < \frac{m(\lambda)}{f^\lambda} < 1 + \varepsilon.$$

Proof: Let P_n be the number of partitions of n. A classical result of Hardy and Ramanujan shows that there exists a constant $c > 1$ such that $c^{\sqrt{n}} > P_n$ for every n. See [A] (5.1.2). The bijection between the conjugacy classes of S_n and the partitions of n (given by the cycle structure) implies that the number of conjugacy classes with support of order s is not more than P_s; so, less than $c^{\sqrt{s}}$.

Let $1 > \varepsilon > 0$ and let

$$\delta = \delta(\varepsilon) = \frac{1}{2c^2}\left(\frac{\varepsilon}{2+\varepsilon}\right)^2.$$

Let $0 < q < 1$ and $0 < b$ be the constants that appear in Theorem 1.2 and let s_0 be the minimal positive integer so that

$$\sum_{s=s_0+1}^{\infty} c^{\sqrt{s}}(\max\{\delta, q\})^{bs} < \frac{\varepsilon}{2}.$$

Lemma 1.1 implies that for every irreducible representation S^λ of S_n

$$\begin{aligned}|m(\lambda) - f^\lambda| &= |\sum_C \chi^\lambda(C) - f^\lambda| \\ &= |\sum_{C \neq \mathrm{id}} \chi^\lambda(C)| \\ &\leq \sum_{\mathrm{supp}(C) > s_0} |\chi^\lambda(C)| + \sum_{0 < \mathrm{supp}(C) \leq s_0} |\chi^\lambda(C)|\end{aligned}$$

where the sums run over all conjugacy classes C in S_n with the prescribed restrictions.

It follows from Theorem 1.2 that

$$\begin{aligned}\sum_{\mathrm{supp}(C)>s_0} |\chi^\lambda(C)| &= f^\lambda \cdot \sum_{\mathrm{supp}(C)>s_0} |r^\lambda(C)| \\ &\le f^\lambda \cdot \sum_{s=s_0+1}^{n} P_s \left(\max\left\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}, q\right\}\right)^{b\cdot s} \\ &\le f^\lambda \cdot \sum_{s=s_0+1}^{n} c^{\sqrt{s}} \left(\max\left\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}, q\right\}\right)^{b\cdot s}.\end{aligned}$$

So, for a partition λ with $\max\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}\} \le \delta$,

$$\sum_{\mathrm{supp}(C)>s_0} |\chi^\lambda(C)| \le f^\lambda \cdot \sum_{s=s_0+1}^{n} c^{\sqrt{s}} (\max\{\delta, q\})^{b\cdot s}.$$

By the definition of s_0 this upper bound is less than $\frac{\varepsilon}{2} f^\lambda$.

Let C be a conjugacy class in S_n with support $\le s_0$ and γ_p cycles of length p. Theorem 1.3 shows that

$$r^\lambda(C) = \prod_{p\ge 2} \left(\sum_{i=1}^{d} [\alpha_i^p - (-\beta_i)^p]\right)^{\gamma_p} + O\left(\frac{1}{n}\right)$$

where

$$\alpha_i = \frac{\lambda_i - i + \frac{1}{2}}{n}, \quad \beta_i = \frac{\lambda_i' - i + \frac{1}{2}}{n}$$

and the constant in $O(\frac{1}{n})$ depends only on C.

For every partition λ of n with $\max\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}\} \le \delta$ and every $p \ge 2$,

$$\begin{aligned}\left|\sum_i (\alpha_i^p - (-\beta_i)^p)\right| &\le \sum_i \left(\left(\frac{\lambda_i}{n}\right)^p + \left(\frac{\lambda_i'}{n}\right)^p\right) \\ &\le \left(\frac{\lambda_1}{n}\right)^{p-1} \sum_i \frac{\lambda_i}{n} + \left(\frac{\lambda_1'}{n}\right)^{p-1} \sum_i \frac{\lambda_i'}{n} \\ &\le 2\delta^{p-1}.\end{aligned}$$

By definition of the support, for every conjugacy class C,

$$\sum_{p\ge 2} \gamma_p \cdot p = \mathrm{supp}(C) \quad \text{and} \quad \sum_{p\ge 2} \gamma_p \le \frac{\mathrm{supp}(C)}{2}.$$

Hence, for every conjugacy class C with support $\le s_0$

$$|r^\lambda(C)| \le \prod_{p\ge 2}(2\delta^{p-1})^{\gamma_p} + O\left(\frac{1}{n}\right) = \delta^{\Sigma\gamma_p\cdot p}\cdot\left(\frac{2}{\delta}\right)^{\Sigma\gamma_p} + O\left(\frac{1}{n}\right)$$
$$\le (2\delta)^{\mathrm{supp}(C)/2} + O\left(\frac{1}{n}\right).$$

The constant in $O(\frac{1}{n})$ depends only on the conjugacy class C and the number of conjugacy classes with $\mathrm{supp}(C) \le s_0$ is bounded. Hence,

$$\sum_{0<\mathrm{supp}(C)\le s_0} |\chi^\lambda(C)| = f^\lambda \cdot \sum_{0<\mathrm{supp}(C)\le s_0} |r^\lambda(C)|$$
$$\le f^\lambda \cdot \sum_{s=1}^{s_0} P_s(2\delta)^{s/2} + O\left(\frac{1}{n}\right)$$
$$\le f^\lambda \cdot \sum_{s=1}^{s_0} c^{\sqrt{s}}(2\delta)^{s/2} + O\left(\frac{1}{n}\right)$$
$$\le f^\lambda \cdot \sum_{s=1}^{s_0} (2\delta c^2)^{s/2} + O\left(\frac{1}{n}\right).$$

Recall that δ was chosen to be $\frac{1}{2c^2}\left(\frac{\varepsilon}{2+\varepsilon}\right)^2$. So, for sufficiently large n the above upper bound is less than $\frac{\varepsilon}{2}f^\lambda$. This completes the proof. ∎

Note that for large n, most irreducible representations of S_n satisfy the conditions of Theorem 2.1. The following is a complementary theorem for other representations.

THEOREM 2.2: *Let $m(\lambda)$ be the multiplicity of the irreducible representation S^λ in the conjugacy representation, and let f^λ be the multiplicity of S^λ in the regular representation.*

For any fixed $1 > r > 0$ and $\varepsilon > 0$ there exist $k = k(\varepsilon, r)$ and $N(\varepsilon, r)$ such that, for any partition λ of $n > N(\varepsilon, r)$ with $\max\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}\} < r$,

$$A - \varepsilon < \frac{m(\lambda)}{f^\lambda} < A + \varepsilon$$

where A is a constant which depends only on the fractions $\frac{\lambda_1}{n}, \dots, \frac{\lambda_k}{n}, \frac{\lambda_1'}{n}, \dots, \frac{\lambda_k'}{n}$.

Proof: Let c and q be as in the previous proof, and s_0 be the minimal positive integer so that $\sum_{s=s_0+1}^{\infty} c^{\sqrt{s}}(\max\{r, q\})^{bs} < \varepsilon/2$.

The calculations carried out in the previous proof show that

$$|m(\lambda) - f^\lambda - f^\lambda \sum_{0<\mathrm{supp}(C)\le s_0} r^\lambda(C)| = |f^\lambda \sum_{\mathrm{supp}(C)>s_0} r^\lambda(C)| \le f^\lambda \cdot \frac{\varepsilon}{2}.$$

Therefore, it suffices to prove that there exists a constant $B = A - 1$ so that

$$\left| \sum_{0<\mathrm{supp}(C)\le s_0} r^\lambda(C) - B \right| \le \frac{\varepsilon}{2}. \tag{1}$$

According to Theorem 1.3

$$\sum_{0<\mathrm{supp}(C)\le s_0} r^\lambda(C) = \sum_{0<\mathrm{supp}(C)\le s_0} \left(\prod_{p\ge 2} \left(\sum_{i=1}^{d} [\alpha_i^p - (-\beta_i)^p] \right)^{\gamma_p} + O\left(\frac{1}{n}\right) \right)$$

where the constant in $O(\frac{1}{n})$ depends only on the conjugacy classes. The sum runs over a bounded number of conjugacy classes. Thus

$$\sum_{0<\mathrm{supp}(C)\le s_0} r^\lambda(C) = \sum_{0<\mathrm{supp}(C)\le s_0} \left(\prod_{p\ge 2} \left(\sum_{i=1}^{d} [\alpha_i^p - (-\beta_i)^p] \right)^{\gamma_p} \right) + O\left(\frac{1}{n}\right) \tag{2}$$

where the constant in $O(\frac{1}{n})$ depends only on s_0.

For any partition λ and any k, $\max\{\lambda_k, \lambda'_k\} \le \frac{n}{k}$. Hence for $d \ge k$ and $p \ge 2$,

$$\left| \sum_{i=1}^{d} [\alpha_i^p - (-\beta_i)^p] - \sum_{i=1}^{k} [\alpha_i^p - (-\beta_i)^p] \right| \le \sum_{i=k+1}^{d} (\alpha_i^p + \beta_i^p) \le \frac{2}{(p-1)k^{p-1}} \le \frac{2}{k^{p-1}}.$$

For any partition λ, all α_i's and β_i's are nonnegative fractions and by Fact 1.4 $\sum_{i=1}^{d}(\alpha_i + \beta_i) = 1$. So, for $p \ge 2$, $|\sum_{i=1}^{k}[\alpha_i^p - (-\beta_i)^p]| \le 1$. Let $\alpha_i = 0$ and $\beta_i = 0$ for $i > d$. We obtain for any d and k

$$\begin{aligned}
&\prod_{p\ge 2} \Big(\sum_{i=1}^{d} [\alpha_i^p - (-\beta_i)^p]\Big)^{\gamma_p} \\
&\le \prod_{p\ge 2} \left(\sum_{i=1}^{k} [\alpha_i^p - (-\beta_i)^p] + \frac{2}{k^{p-1}} \right)^{\gamma_p} \\
&= \sum_{i_1,\ldots,i_p} \prod_{p\ge 2} \binom{\gamma_p}{i_p} [\alpha_i^p - (-\beta_i)^p]^{i_p} \left(\frac{2}{k^{p-1}} \right)^{\gamma_p - i_p}
\end{aligned}$$

$$= \prod_{p\ge 2}[\alpha_i^p - (-\beta_i)^p]^{\gamma_p} + \sum_{\exists j, i_j\ne\gamma_j}\prod_{p\ge 2}\binom{\gamma_p}{i_p}[\alpha_i^p-(-\beta_i)^p]^{i_p}\left(\frac{2}{k^{p-1}}\right)^{\gamma_p-i_p}$$

$$\le \prod_{p\ge 2}[\alpha_i^p - (-\beta_i)^p]^{\gamma_p} + \sum_{\exists j, i_j\ne\gamma_j}\prod_{p\ge 2}\binom{\gamma_p}{i_p}\left(\frac{2}{k^{p-1}}\right)^{\gamma_p-i_p}$$

$$\le \prod_{p\ge 2}\left(\sum_{i=1}^{k}[\alpha_i^p-(-\beta_i)^p]\right)^{\gamma_p} + \prod_{p\ge 2}\left(1+\frac{2}{k^{p-1}}\right)^{\gamma_p} - 1;$$

$\Sigma\gamma_p = \mathrm{supp}(C) \le s_0$. So, for $2/k \le 1$,

$$\prod_{p\ge 2}\left(1+\frac{2}{k^{p-1}}\right)^{\gamma_p} - 1 \le \left(1+\frac{2}{k}\right)^{s_0} - 1 \le \sum_{t=1}^{s_0}\binom{s_0}{t}\left(\frac{2}{k}\right)^t \le \frac{2}{k}\sum_{t=1}^{s_0}\binom{s_0}{t} \le \frac{2^{s_0+1}}{k}$$

and

$$\prod_{p\ge 2}\left(\sum_{i=1}^{d}[\alpha_i^p-(-\beta_i)^p]\right)^{\gamma_p} \le \prod_{p\ge 2}\left(\sum_{i=1}^{k}[\alpha_i^p-(-\beta_i)^p]\right)^{\gamma_p} + \frac{2^{s_0+1}}{k}.$$

Similarly,

$$\prod_{p\ge 2}\left(\sum_{i=1}^{k}[\alpha_i^p-(-\beta_i)^p]\right)^{\gamma_p} - \frac{2^{s_0+1}}{k} \le \prod_{p\ge 2}\left(\sum_{i=1}^{d}[\alpha_i^p-(-\beta_i)^p]\right)^{\gamma_p}.$$

The number of the conjugacy classes in (2) is bounded. We conclude

$$\begin{aligned}(3)\qquad &\sum_{0<\mathrm{supp}(C)\le s_0}\prod_{p\ge 2}\left(\sum_{i=1}^{d}[\alpha_i^p-(-\beta_i)^p]\right)^{\gamma_p}\\ &-\sum_{0<\mathrm{supp}(C)\le s_0}\prod_{p\ge 2}\left(\sum_{i=1}^{k}[\alpha_i^p-(-\beta_i)^p]\right)^{\gamma_p} = O\left(\frac{1}{k}\right),\end{aligned}$$

where the constant in $O(\frac{1}{k})$ depends only on s_0.

On the other hand

$$\left|\sum_{i=1}^{k}\left[\left(\frac{\lambda_i}{n}\right)^p - \left(-\frac{\lambda_i'}{n}\right)^p\right] - \sum_{i=1}^{k}[\alpha_i^p-(-\beta_i)^p]\right|$$

$$\le \sum_{i=1}^{k}\left[\left(\alpha_i+\frac{i}{n}\right)^p - \alpha_i^p + \left(\beta_i+\frac{i}{n}\right)^p - \beta_i^p\right]$$

$$= \sum_{i=1}^{k}\sum_{t=0}^{p-1}\binom{p}{t}\left[\alpha_i^t\left(\frac{i}{n}\right)^{p-t} + \beta_i^t\left(\frac{i}{n}\right)^{p-t}\right]$$

$$\le 2\sum_{t=0}^{p-1}\binom{p}{t}\sum_{i=1}^{k}\frac{i}{n} \le \frac{2^p(k+1)^2}{n}.$$

Hence (by calculations as above)

$$\begin{aligned}(4)\qquad &\Bigg| \sum_{0<\operatorname{supp}(C)\le s_0} \prod_{p\ge 2} \left(\sum_{i=1}^{k} \left[\left(\frac{\lambda_i}{n}\right)^p - \left(-\frac{\lambda_i'}{n}\right)^p \right] \right)^{\gamma_p} \\ &- \sum_{0<\operatorname{supp}(C)\le s_0} \prod_{p\ge 2} \left(\sum_{i=1}^{k} [\alpha_i^p - (-\beta_i)^p] \right)^{\gamma_p} \Bigg| \\ &\le O\left(\frac{k^2}{n}\right)\end{aligned}$$

where the constant in $O(k^2/n)$ depends only on s_0.

Combining (1)–(4), and setting $k = \Omega(1/\varepsilon)$ and $N(\varepsilon, r) = \Omega(k^2/\varepsilon)$, where the constants in Ω depend only on s_0, we are done with

$$\begin{aligned} A &= A\left(\frac{\lambda_1}{n}, \ldots, \frac{\lambda_k}{n}, \frac{\lambda_1'}{n}, \ldots, \frac{\lambda_k'}{n}\right) \\ &= 1 + \sum_{0<\operatorname{supp}(C)\le s_0} \prod_{p\ge 2} \left(\sum_{i=1}^{k} \left[\left(\frac{\lambda_i}{n}\right)^p - \left(-\frac{\lambda_i'}{n}\right)^p \right] \right)^{\gamma_p}. \qquad \blacksquare \end{aligned}$$

The proof implies that the constant A is not necessarily equal to 1. We conclude that if $\max\{\lambda_1, \lambda_1'\}$ is proportional to n, then the multiplicity $m(\lambda)$ is proportional to f^λ but not necessarily "close" to f^λ.

If $\max\{\frac{\lambda_1}{n}, \frac{\lambda_1'}{n}\}$ is not bounded away from 1, then Theorem 2.2 does not hold and the multiplicity is not sufficiently proportional to f^λ. (For example, the multiplicity of the trivial representation equals the number of conjugacy classes.) We leave this case open.

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References

[AF] R. M. Adin and A. Frumkin, *The conjugacy character tends to be regular*, Israel Journal of Mathematics **59** (1987), 234–240.

[A] George E. Andrews, *The Theory of Partitions*, in *Encyclopedia of Mathematics and its Applications*, Vol. 2, Addison-Wesley, 1976.

[FOW] L. Flatto, A. M. Odlyzko and D. B. Wales, *Random shuffles and group representations*, The Annals of Probability **13** (1985), 154–178.

[Fo] E. Formanek, *The conjugation representation and fusionless extensions*, Proceedings of the American Mathematical Society **30** (1971), 73–74.

[Fr] J. S. Frame, *On the reduction of the conjugating representation of a finite group*, Bulletin of the American Mathematical Society **53** (1948), 584–589.

[F] A. Frumkin, *Theorem about the conjugacy representation of S_n*, Israel Journal of Mathematics **55** (1986), 121–128.

[K] E. Kaniuth, *Irreducible subrepresentations of the conjugation representation of finite p groups*, Manuscripta Mathematica **74** (1992), 161–175.

[L1] D. N. Li, *Properties of representation matrices of finite groups*, Bulletin of the Korean Academy of Sciences **27** (1979), 1–3 (in Korean).

[L2] D. N. Li, *Conditions for the conjugacy representation of a finite group to contain all irreducible representations*, Korean Research Center for Physics and Mathematics **2** (1980), 21–31 (in Korean).

[R] Y. Roichman, *Cayley graphs of the symmetric groups*, Ph.D. thesis, Hebrew University, 1994 (in Hebrew).

[R1] Y. Roichman, *Upper bound on the characters of the symmetric groups*, to appear in Inventiones Mathematicae.

[Rt] R. L. Roth, *On the conjugating representation of a finite group*, Pacific Journal of Mathematics **36** (1971), 515–521.

[Sa] J. Saxl, personal communication.

[Sc] T. Scharf, *Ein weiterer Beweis, das die konjugierende der symmetrischen Gruppen jede irreduzible Darstellung enhalt*, Archiv der Mathematik **54** (1990), 427–429.

[Se] J. P. Serre, *Linear Representations of Finite Groups*, Springer-Verlag, 1977.

[So] L. Solomon, *On the sum of the elements in the character table of a finite group*, Proceedings of the American Mathematical Society **12** (1961), 962–963.

[St] R. Stanley, personal communication.

[Wa] A. J. Wasserman, *Automorphic actions of compact groups on operator algebras*, Ph.D. thesis, University of Pennsylvania, 1981. See also [FOW] pp. 161–163.